# Exercise 37

Prove that $\lim_{x\to a}\sqrt{x}=\sqrt{a}$ if $a>0$.

$$\left[\textit{Hint: } \text{Use } |\sqrt{x}-\sqrt{a}\,| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}.\right]$$

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**Solution**

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if}\quad 0<|x-a|<\delta \qquad \text{then} \qquad \left|\sqrt{x}-\sqrt{a}\right|<\varepsilon$$

for all positive $\varepsilon$. Start by working backwards, looking for a number $\delta$ that's greater than $|x-a|$.

$$\left|\sqrt{x}-\sqrt{a}\right|<\varepsilon$$

$$\frac{|x-a|}{\sqrt{x}+\sqrt{a}}<\varepsilon$$

$$\left(\frac{1}{\sqrt{x}+\sqrt{a}}\right)|x-a|<\varepsilon$$

On an interval centered at $x=a$, a positive constant $C$ can be chosen so that $1/(\sqrt{x}+\sqrt{a})<C$.

$$C|x-a|<\varepsilon$$

$$|x-a|<\frac{\varepsilon}{C}$$

To determine $C$, suppose that $x$ is within a distance $b$ from $a$.

$$|x-b|<a$$

$$-a<x-b<a$$

$$b-a<x<b+a$$

$$\sqrt{b-a}<\sqrt{x}<\sqrt{b+a}$$

$$\sqrt{b-a}+\sqrt{a}<\sqrt{x}+\sqrt{a}<\sqrt{b+a}+\sqrt{a}$$

$$\frac{1}{\sqrt{b-a}+\sqrt{a}}>\frac{1}{\sqrt{x}+\sqrt{a}}>\frac{1}{\sqrt{b+a}+\sqrt{a}}$$

The constant $C$ is then $1/(\sqrt{b-a}+\sqrt{a})$, which means $\varepsilon/C=(\sqrt{b-a}+\sqrt{a})\varepsilon$.

Choose $\delta$ to be whichever is smaller between $a$ and $(\sqrt{b-a}+\sqrt{a})\varepsilon$: $\delta = \min\{a, (\sqrt{b-a}+\sqrt{a})\varepsilon\}$. Now, assuming that $|x-a| < \delta$,

$$\begin{aligned}
\left|\sqrt{x}-\sqrt{a}\right| &= \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \\
&= \left(\frac{1}{\sqrt{x}+\sqrt{a}}\right)|x-a| \\
&< C\delta \\
&= \left(\frac{1}{\sqrt{b-a}+\sqrt{a}}\right)\left[(\sqrt{b-a}+\sqrt{a})\varepsilon\right] \\
&= \varepsilon.
\end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to a}\sqrt{x} = \sqrt{a}$$

if $a > 0$.